Problem Set 8
Optical Waveguides and Fibers (OWF)

will be discussed in the tutorial on January 13, 2016

Problem 1: Linearly polarized modes (LP) in a step-index fiber.

The modes of a step-index fiber can be calculated analytically in an *exact* form, leading to a classification in $\mathrm{TE}_{0,\mu}$, $\mathrm{TM}_{0,\mu}$ and hybrid modes ($EH_{\nu,\mu}$ and $HE_{\nu,\mu}$). When looking for exact solutions, one can find a differential equation for the $\underline{\mathcal{E}}_z$ and $\underline{\mathcal{H}}_z$ components, from which the transverse components can be derived. A simplified approximation can be used under the assumption that the mode is weakly guided ($n_1 \to n_2$) and has a dominant linearly polarized transverse field component, which without loss of generality we denote as $\underline{\mathcal{E}}_x$, while assuming $\underline{\mathcal{E}}_y = 0$.

Because of the assumption of weak guidance, the scalar Helmholtz equation can be used:

$$\nabla^2 \underline{\Psi}(r,\varphi) + \left(k_0^2 n^2 - \beta^2\right) \underline{\Psi}(r,\varphi) = 0, \tag{1}$$

where $\underline{\Psi}(r,\varphi)$ denotes the $\underline{\mathcal{E}}_x$ component of the mode.

a) Write Eq. (1) in cylindrical coordinates.

b) Separate the variables, i.e., assume that the solution can be written in the form $\underline{\Psi}(r,\varphi) = g(r)h(\varphi)$. Insert this ansatz into the result from part a), separate it into a sum of two expressions where one depends exclusively on r and the other exclusively on φ. Show that $\sin(\nu\varphi)$ and $\cos(\nu\varphi)$ are solutions for the φ-dependent part. Why must ν be an integer?

c) Insert the sinusoidal solution for $h(\varphi)$ into the result of part a) and show that the differential equation for $g(r)$ can be written as:

$$r^2 \frac{\partial^2 g(r)}{\partial r^2} + r \frac{\partial g(r)}{\partial r} + \left[\left(k_0^2 n_i^2 - \beta^2\right) r^2 - \nu^2\right] g(r) = 0, \tag{2}$$

where n_1 is the core index and n_2 is the cladding index.

Using the fact that Eq. (2) is solved by Bessel functions and modified Bessel functions, the total solution of Eq. (1) can be written as:

$$\underline{\Psi}(r,\varphi) = \begin{cases} A \mathrm{J}_\nu\left(u\frac{r}{a}\right)\cos(\nu\varphi+\psi) & \text{for } 0 \le x \le a \\ A\frac{\mathrm{J}_\nu(u)}{\mathrm{K}_\nu(w)} \mathrm{K}_\nu\left(w\frac{r}{a}\right)\cos(\nu\varphi+\psi) & \text{for } a < x \end{cases} \tag{3}$$

where J_ν is the Bessel function of the first kind of order ν, K_ν is the decaying modified Bessel function of order $\nu = 0, 1, 2, ...$, $\psi \in \{0, \frac{\pi}{2}\}$, $u = a\sqrt{k_0^2 n_1^2 - \beta^2}$, $w = a\sqrt{\beta^2 - k_0^2 n_2^2}$.
In this relation we assumed that $\underline{\Psi}(r,\varphi)$ is continuous at $r = a$.

d) Why is this assumption legitimate?

Starting from the equation

$$\nabla \cdot \underline{\mathbf{D}} = 0 \tag{4}$$

it is possible to show that in the limit $n_1 \to n_2$ the derivative $\frac{\partial \Psi}{\partial r}$ must be continuous as well.

e) Use this fact to derive the characteristic equation for LP-modes:

$$\frac{u\mathrm{J}'_\nu(u)}{\mathrm{J}_\nu(u)} = \frac{w\mathrm{K}'_\nu(w)}{\mathrm{K}_\nu(w)} \tag{5}$$

f) We want now to simplify Eq. (5) by getting rid of the derivative of the Bessel function. For this purpose, make use of the recursive relations,

$$\mathrm{J}'_\nu(u) = +\mathrm{J}_{\nu-1}(u) - \frac{\nu}{u}\mathrm{J}_\nu(u) \quad , \tag{6}$$

$$\mathrm{K}'_\nu(w) = -\mathrm{K}_{\nu-1}(w) - \frac{\nu}{w}\mathrm{K}_\nu(w) \quad , \tag{7}$$

and show that Eq. (5) implies:

$$\frac{u\mathrm{J}_{\nu-1}(u)}{\mathrm{J}_\nu(u)} = -\frac{w\mathrm{K}_{\nu-1}(w)}{\mathrm{K}_v(w)} \tag{8}$$

For each index ν the latter equation can be solved for β, as done already for the slab waveguide. Since the Bessel function oscillates, different solutions are obtained and can be classified by means of a new integer, μ. The normalized cut-off frequencies $V_{\mu,\nu,c}$ of the different modes are obtained from Eq. (8) when we set $w \to 0$ (and simultaneously $u \to V = ak_0\sqrt{n_1^2 - n_2^2}$). From standard properties of the Bessel functions, it can be proven that $\lim_{w\to 0} \frac{w\mathrm{K}_{\nu-1}(w)}{\mathrm{K}_\nu(w)} = 0$. The normalized cut-off frequency of the $\mathrm{LP}_{\nu,\mu}$ mode ($\mu = 1, 2, 3...$) is hence determined by the μ−th zero $j_{\nu-1,\mu}$ of the Bessel function $\mathrm{J}_{\nu-1}(u)$.

$$V_{\mu,\nu,\mathrm{c}} = j_{\nu-1,\mu} \tag{9}$$

g) A typical standard single mode fiber has the following specifications: $a = 4.1\,\mu$m, $\Delta = \frac{n_1^2 - n_2^2}{2n_1^2} = 0.0035$ and $n_1 = 1.41$. This fiber always supports the fundamental mode $\mathrm{LP}_{0,1}$. The next higher order mode is the $\mathrm{LP}_{1,1}$. What is the minimum wavelength for which the fiber is single-mode? Hint: $j_{0,1} \approx 2.4048$.

Questions and Comments:

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